Total GMAT Math (Excerpt)

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1 Introduction

Over the last twelve years, I've taught dozens of classes and hundreds of students, all with the aim of helping people get the highest GMAT score they possibly can. During that time, I've worked with materials from large test-prep companies, questions published by the test-maker, and a variety of supplements that I've devised myself. But still, despite the proliferation of test-prep materials on the market, people still come to me asking if there's something else out there, a single resource that contains everything an aspiring MBA needs to know to ace the Quantitative portion of the GMAT.

Now there is. Total GMAT Math is that single comprehensive resource. In this book, you'll find tutorials on content and strategy, plus tidbits on time management and weeks worth of relevant practical material. You may well want to seek out additional material (and this book will tell you where to look), but you don't have to.

Unlike many other GMAT math books, this resource presumes almost no math knowledge. I start from the basics of manipulating fractions and simple algebraic equations, and steadily work up to the most complex content on the test. There may be times when you feel the need to read over a section three or four times (in fact, I encourage it!) but you'll never be forced to consult another book just to get through a section of this one.

My favorite part of this guide, especially when comparing it to the other options out there, is the combination of content and practice. For each of about forty sections, covering everything from mental math to simultaneous rate calculations to sequences, you'll have three different perspectives on the material:

First, an exposition. This explains the principles behind the content, the basic facts and techniques you need, and the ways in which the material will come up in test questions.

Second, a series of exercises. These questions are not in the GMAT format, but are designed to make sure that you understood the techniques that you read about in the first part.

Finally, a series of GMAT-like practice questions. There are between 5 and 10 practice items per section, totalling 300 problems. In most sections, some questions are identified as "Challenge" questions. These are more difficult and take you a step beyond the other practice questions. As you might expect, the easier content areas have no challenge questions, while the more difficult ones have almost all challenge questions.

The benefits of this structure, I think, are obvious. Unlike most books, in which you do a random sequence of questions every day, learning a technique then forgetting about it for a week, this one gives you test-like review for every single concept that the GMAT math section tests. Every one! While this book doesn't contain every single possible question that could arise on the GMAT, it does have a representative cross-section. When you finish each content area, you'll be well-equipped to take on those questions the next time you see them.

As you can probably tell, I'm very excited to release this book into the world. If it already existed, I would be using it with all of my students; heck, some of my students could probably have gotten by with only the book. With a well-structured, thorough resource like this one, you won't need to rely so much on classes or tutoring. Sticking to a steady preperation schedule is enough.

2 Word Problems: Rate

Like ratios, it's useful to think of rates as fractions. In fact, rates are essentially the same as ratios, with one key difference. Ratios express the relationship between two like things: if the ratio of men to women is 3 : 4, we may separate people into two segments, but they are all people. Rates, on the other hand, express the relationship between two unlike things.

The most common type of rate is speed. You've probably heard the phrase "miles per hour" or "kilometers per hour" so many times in your life you've long since stopped thinking about it as a relationship between two different quantities. But a relationship is, in fact, what it is. When you say you're traveling at 60 miles per hour, you're expressing a relationship between the number of miles you're traveling and the number of hours you're traveling:

$\frac{60miles}{1hour}$

You can handle many of the rate problems you'll see on the GMAT if you treat them just like the ratios you saw a few chapters ago. Just as with ratios, you'll be given a relationship ("60 miles per hour") and one actual number (say, 180 miles). With those, you can set up a ratio:

$\frac{60miles}{1hour} = \frac{180miles}{xhours}$

Since the units are the same on both sides of the equation, you can forget about them. Then, cross-multiply and solve for x:

$60x = 180(1)$

$x = \frac{180}{60} = 3$

Rate problems need not be speed-related, though. You will see plenty of rates that express some form of speed, such as:

miles per hour
widgets per day
dollars per month

However, a rate can consist of any relationship of two unlike things. Here are some examples of rates you might come across on GMAT questions:

sales per customer
GDP per capita
shares of stock per portfolio

While you may be most comfortable with speed-related rates, it's important to understand rates at a more abstract level, one that allows you to accomodate these other types of rates with ease. To reach that level, force yourself to express every single rate you see as a fraction, consciously using the word "per" as a signal. Just as "miles per hour" means you'll have some number of miles divided by some number of hours, "GDP per capita" means you'll have some amount of GDP divided by some number of people.

In most such cases, the problem isn't much more complicated than anything you'd see on a ratio question. The techniques discussed in that section work just as well here.

GMAT rate questions get tricky when the test combines multiple rates. In fact, one type of question, sometimes called "work" or "simultaneous rate," merits the entire next chapter. Before getting to work problems, we'll take a look at a couple of others way in which the GMAT will combine rates.

The first type of problem is "average rate" or "average speed." An example might go like this:

> "Karen drove 100 miles at a speed of 40 miles per hour, then another 100 miles at a speed of 50 miles per hour. What was her average speed for the entire 200-mile trip?"

Most people, upon initial exposure to this type of question, have a reflex answer: 45 miles per hour. Accordingly, most people are wrong. To discover why, we'll need to walk through the problem, step by step.

The formula for average speed is just like that for any other type of rate. If we're looking for miles per hour, we need to find the total miles over the total number of hours. Total miles in this case is easy: 200 miles. Total hours takes more effort. 100 miles at 40 miles per hour takes 2.5 hours, while 100 miles at 50 miles per hour takes 2 hours. Thus, total hours is 4.5, and the average speed is:

$$\frac{200}{4.5} = 44\frac{4}{9}$$

Why do we get such a counter-intuitive answer? The key is the times that we solved for. Karen spent more time driving at the slower speed, so her average speed was closer to 40 than 50. The average speed is weighted by amounts of time, so the fact that the distances were the same doesn't mean the weights are equal. (For more on weighted averages, consult the chapter specifically covering that topic.)

It's great if you understand exactly why the intuitive answer is wrong. But most importantly, you need to remember that to solve for an average speed, you must calculate total distance and total time, as we did working through this example.

The other type of combined rate problem involves two objects moving toward each other. (Or, in another variation, one catching up with the other.) The stereotypical example goes something like this:

> "Stations A and B are 100 miles apart. If Train X leaves Station A moving at 20 miles per hour and Train Y leaves Station B at the same time moving at 30 miles per hour along a parallel track, how long will the trains travel before they meet?"

The key concept needed to solve a problem like this is that the trains move toward each other at the sum of their rates. After one hour, Train X will have traveled 20 miles, and Train Y will have traveled 30 miles. So after that first hour, the trains are 50 miles closer to each other than they were when they started. Another way to put that is to say that the trains are converging at 50 miles per hour.

The math involved is very simple. In fact, it's the same math you'd need to figure out how long it takes you to drive 100 miles at a speed of 50 miles per hour. It's 2 hours. If the question asked, instead, how far Train Y traveled before the trains meet, you would begin the problem the same way. Once you discovered that it took 2 hours before they met, you'd use Train Y's speed (30 miles per hour) to determine how far Train Y traveled in 2 hours: 60 miles.

The variation–one object catching up with another–is very similar. Here's an example of the basic framework that these questions follow:

> "Ron and Sara are driving along the same road. Sara is driving at a constant rate of 30 miles per hour, while Ron is driving at a constant rate of 40 miles per hour. If Ron is currently 5 miles behind Sara, at what point will Ron catch up with Sara?"

Much like the converging trains of the previous example, the first step here is to distill the two rates into a single number. We don't particularly care how far Ron or Sara get, or how fast they do so. What matters is how fast Ron catches up. We can find that by subtracting Sara's rate from Ron's rate, for a "catch-up rate" of 10 miles per hour. In an hour, Ron drives 10 more miles than Sara does, so that's the number of miles Ron gains on Sara.

The final step is identical to that of the train example above. We want to know when Ron will make up 5 miles, and he makes up 10 miles per hour. Given that information, Ron will catch up in a half hour.

3 Word Problems: Rate: Drill

1. In 8 hours, a factory produces 176,000 wing nuts. How many wing nuts does the factory produce per hour?
2. An airplane travels a 1,200 mile route in 4 hours. What is the airplane's average speed for the trip?
3. A certain insect can crawl 900 feet in 12 hours. What is the insect's average distance per hour over that time?
4. The national income of Country X is $1,800,000,000. If Country X has a population of 3 million, what is its per capita national income?
5. If Charles drives at an average speed of 55 miles per hour, how long would it take him to drive 275 miles?
6. A certain hose fills a tank at a rate of 16 gallons per minute. If the tank has a capacity of 960 gallons, how long would it take the hose to fill the tank?
7. How long would it take someone walking at a rate of 4 miles per hour to cover a distance of 27 miles?
8. If Oscar can build 24 feet of fencing per day, how many days would it take Oscar to build a 144 foot long fence?
9. A train travels at an average rate of 35 miles per hour for 2.5 hours. What distance does it cover in that time?
10. A certain university graduates 120 students per year. How many students does it graduate in an 8-year period?
11. A pool drains at a rate of 600 gallons of water per hour. How many gallons of water will be drained from the pool in $1\frac{1}{4}$ hours?
12. How many meters will a satellite travel in 1 minute if it travels at an average speed of 18 meters per second?
13. If Eileen drives at an average speed of 50 miles per hour for 3 hours and an average speed of 60 miles per hour for the next 2 hours, what is her average speed for the entire trip?
14. A train travels at a speed of 60 miles per hour for the first 180 miles of its trip, then it travels at a speed of 45 miles per hour for the remaining 180 miles of its trip. What is the train's average speed for the entire trip?
15. A factory produces 220 engines per week for six weeks. Following the installation of new equipment, the factory produces 320 engines per week for next eight weeks. What is the average number of engines produced per week over the 14-week period?
16. If a truck driver travels 455 miles at an average speed of 70 miles per hour, then takes a one-hour break, then travels another 315 miles at the same speed, what is his average speed for the entire 770-mile trip, including the break?

17. Ana and Bradley are 12 miles away from each other on opposite ends of a straight path. If Ana walks toward Bradley at a rate of 4 miles per hour and Bradley walks toward Ana at a rate of 5 miles per hour, how long will it take before they meet?
18. Xavier and Yoelle are proofreading a 144-page manuscript. Xavier starts at the first page and works at a rate of 20 pages per hour and Yoelle starts at the last page and works at a rate of 16 pages per hour. When Xavier and Yoelle have read the entire manuscript between them, how many pages will Xavier have read?
19. Frankie and Georgia are driving along the same road. If Frankie is driving at a speed of 52 miles per hour and Georgia is 30 miles behind him, driving at a speed of 58 miles per hour, how long will it take before Georgia catches up with Frankie?
20. Janelle and Karl both sell insurance. This year, Janelle has sold $60,000 less than Karl has. If Janelle sells $3,200 per day and Karl sells $2,000 per day, how many days will it take before Janelle catches up with Karl and moves $12,000 ahead?

4 Word Problems: Rate: Practice

201. How long did it take Fiona to drive the 240 miles nonstop from her home to Columbus, Ohio?

(1) Fiona's average speed over the first 180 miles was 50 miles per hour.

(2) Fiona's average speed over the final 60 miles was 45 miles per hour.

202. Train X and train Y traveled the same 100-mile route. If train X took 2.5 hours and train Y traveled at an average speed that was 25 percent faster than the average speed of train X, how many hours did it take train Y to travel the route?

(A) 1
(B) $1\frac{3}{4}$
(C) $1\frac{7}{8}$
(D) 2
(E) $3\frac{1}{8}$

203. When flying between Chicago and New York, did a certain airplane ever exceed 400 miles per hour?

(1) By the route the airplane took, the distance between Chicago and New York is 750 miles.

(2) The total time the airplane spent in the air was 2 hours.

204. A rectangular swimming pool with uniform depth is full to capacity with water. If water is pumped out of the pool at a constant rate, how long will it take to empty the pool?

(1) 40 minutes after water begins to be pumped out of the pool, the pool will be $\frac{5}{6}$ full.

(2) 2 hours after water beings to be pumped out of the pool, $\frac{1}{2}$ of the water will have been removed.

205. Factory A fulfills 100 orders twice as fast as factory B does. Factory B fulfills 100 orders in 30 minutes. If each factory fulfills orders at a constant rate, how many orders does factory A fulfill in 9 minutes?

(A) 15
(B) 30
(C) 60
(D) 90
(E) 100

206. A computer printer manufacturer produces 500 units of a certain model each month at a cost to the manufacturer of $65 per unit and all of the produced units are sold each month. What is the minimum selling price per unit that will ensure that the monthly profit (revenue from sales minus the manufacturer's cost to produce) on the sales of these units will be at least $35,000?

(A) $70
(B) $85
(C) $105
(D) $115
(E) $135

207. How long, in minutes, did it take a bicycle wheel to roll along a flat, straight 300-meter path?

(1) The diameter of the bicycle wheel was 0.6 meter.
(2) The wheel made one full 360-degree rotation every 0.6π meters.

208. A phone company charges $\frac{3}{4}$ of its regular per-minute rate for long-distance calls for each minute in excess of 1000 per month, excluding calls made on Sundays, and $\frac{1}{2}$ of its regular per-minute rate for all long-distance calls made on Sundays. How much did the phone company charge Victoria last month?

(A) Last month, Victoria made a total of 1500 minutes worth of long-distance calls, including 200 minutes worth on Sundays.
(B) The regular rate for long-distance calls is $0.10 per minute.

5 Word Problems: Rates: Challenge

209. A shipping company currently charges the same price for each package that it ships. If the current price of each package were to be increased by $1, 6 fewer of the packages could be shipped for $120, excluding sales tax. What is the current price of shipping each package?

(A) $2
(B) $3
(C) $4
(D) $5
(E) $6

210. At the rate of f feet per s seconds, how many feet does a pedestrian travel in m minutes?

(A) $\frac{sm}{f}$
(B) $\frac{fm}{s}$
(C) $\frac{60f}{ms}$
(D) $\frac{60fm}{s}$
(E) $\frac{60fs}{m}$

6 Explanations: Rate: Drill

1. $\frac{176{,}000}{8} = 22,000\frac{wn}{h}$
2. $\frac{1200}{4} = 300\frac{m}{h}$
3. $\frac{900}{12} = 75\frac{f}{h}$
4. $\frac{1{,}800{,}000{,}000}{3{,}000{,}000} = \frac{1{,}800}{3} = 600\frac{\$}{person}$
5. $55 = \frac{275}{t}$
   $55t = 275$
   $t = 5$
6. $16 = \frac{960}{t}$
   $16t = 960$
   $t = \frac{960}{16} = 60$
7. $4 = \frac{27}{t}$
   $4t = 27$
   $t = \frac{27}{4} = 6\frac{3}{4}$
8. $24 = \frac{144}{t}$
   $24t = 144$
   $t = \frac{144}{24} = 6$
9. $35 = \frac{d}{2.5}$
   $d = 35(2.5) = 87.5$
10. $120 = \frac{s}{8}$
    $s = 120(8) = 960$
11. $600 = \frac{g}{1\frac{1}{4}}$
    $g = 600(\frac{5}{4}) = 150(5) = 750$
12. 1 minute = 60 seconds
    $18 = \frac{d}{60}$
    $d = 18(60) = 1080$
13. The distance for the first three hours is $3(50) = 150$, and the distance for the final two hours is $60(2) = 120$. Total distance over total time, then, is:
    $\frac{150+120}{5} = \frac{270}{5} = 54$
14. The first 180 miles take $\frac{180}{60} = 3$ hours, while the second 180 miles take $\frac{180}{45} = 4$ hours. Total distance over total time is:
    $\frac{180+180}{4+3} = \frac{360}{7} = 51\frac{3}{7}$
15. Total engines for the first six weeks is $220(6) = 1320$, and total engines for the last eight weeks is $8(320) = 2560$. Total engines divided by total weeks is:
    $\frac{2560+1320}{6+8} = \frac{3880}{14} = 277\frac{1}{7}$

16. The total time spent on the road is $\frac{770}{70} = 11$ hours, which means that, including the break, his total time is 12 hours. Average speed, then, is:

$\frac{770}{12} = 64\frac{1}{6}$

17. Their combined rate is $4 + 5 = 9$ miles per hour. Thus, we want to find how long it takes to cover 12 miles at 9 miles per hour:

$9 = \frac{12}{x}$

$9x = 12$

$x = \frac{12}{9} = \frac{4}{3} hours$

18. Their combined rate is 36 pages per hour. Thus, it takes them $\frac{144}{36} = 4$ hours to read the whole thing. Since Xavier is reading 20 pages per hour, he'll read a total of $4(20) = 80$ pages.

19. Georgia is driving 6 miles per hour faster, so she's catching up at that rate. To cover 30 miles at 6 miles per hour, it will take $\frac{30}{6} = 5$ hours.

20. Janelle sells $1,200 per day more. To catch up, she needs to sell $60,000 more than Karl, and to get $12,000 ahead, she needs to sell a total of $72,000 more. To sell $72,000 more at a rate of $1,200 more per day, it will require $\frac{72{,}000}{1{,}200} = \frac{720}{12} = 60$ days.

7 Explanations: Rate: Practice

201. C

EXPL: Statement (1) is insufficient: it tells us nothing about the final 60 miles of the trip. Statement (2) has a similar problem: it tells nothing about what preceded the final 60 miles.

Taken together, the statements are sufficient. From (1), we can find the total time for the first 180 miles. From (2), we can find the total time for the final 60 miles. Add those together, and you have the total time for the trip. Choice (C) is correct.

202. D

EXPL: To find train Y's time, we need train Y's speed. We have a relationship between X's and Y's speed, but to use that, we need train X's speed. That we can find, since we're given X's distance and time:

$r = \frac{d}{t} = \frac{100}{2.5} = 40$

25% faster than 40 miles per hour is:

$40(1.25) = 50$

Finally, we can solve for Y's time:

$50 = \frac{100}{t}$

$50t = 100$

$t = 2$, choice (D).

203. E

EXPL: To find the average speed, you need both distance and time. However, even average speed isn't enough to answer this question: if the average speed is, say, 300 miles per hour, that doesn't mean that the plane traveled at 500 miles an hour for a little while and 100 miles an hour for a little bit.

Statements (1) and (2) are both insufficient: neither gives us both pieces of information we need for average speed, let alone enough to determine whether the plane ever exceeded 400 miles per hour.

Taken together, the statements are still insufficient. We can calculate the average speed of 375 miles per hour, but we don't know whether the speed was always 375 miles per hour, or if it fluctuated enough throughout the flight that it surpassed 400 miles per hour at some point. (E) is the correct choice.

204. D

EXPL: Statement (1) is sufficient: if the pool is $\frac{5}{6}$ full after 40 minutes, that means $\frac{1}{6}$ of the water has been pumped out. If it takes 40 minutes to pump out $\frac{1}{6}$ of the water, it takes $40(6) = 240$ minutes to empty the entire pool.

Statement (2) is also sufficient: this time, we learn that $\frac{1}{2}$ the water has been pumped out (leaving $\frac{1}{2}$ the water remaining) in two hours. Thus, it takes $2(2) = 4$ hours to empty the pool. (D) is the correct choice.

205. C

EXPL: Factory B's rate is $\frac{100}{30}$ orders per minute. If A's rate is twice as fast, double the rate: $(\frac{100}{30})2 = \frac{100}{15}$. The number of orders fulfilled in 9 minutes is the product of the rate and the time:

$orders = rt = (\frac{100}{15})(9) = \frac{100}{5}(3) = 20(3) = 60$, choice (C).

206. E

Explanation: In order for the profit to be exactly zero, the selling price per unit would have to be $65: just enough to cover costs.

To determine the price at this profit level, we need to figure out how much profit must come from each unit. Since there are 500 units sold, the desired profit per unit is $\frac{35,000}{500} = 70$.

Since it takes $65 to cover costs and $70 per unit is desired in profit, the minimum selling price to achieve that profit level is $65 + 70 = 135$, choice (E).

207. E

Explanation: To find time, you generally need distance and rate. Distance is given in the question, so you're looking for rate. Statement (1) is insufficient: it gives no clue as to how fast the bicycle wheel was going. Statement (2) is also insufficient: it provides information about the size of the wheel, but nothing about how fast it's going.

Taken together, the statements are insufficient. In fact, the two statements say the same thing: if you know the diameter of the wheel is 0.6, that means the circumference is 0.6π, which is the same as how long it takes for the wheel to make one full rotation. You don't need to get into that much detail to realize that you need the rate, and neither statement gives you that. Choice (E) is correct.

208. C

Explanation: To find what the phone company charged Victoria last month, you need to know how many minutes she used on days other than Sunday, how many minutes she used on Sundays, and what the regular per-minute rate is.

Statement (1) is insufficient: it provides a lot of information about how many minutes she used, but nothing about the per-minute rate.

Statement (2) is also insufficient: it gives you the per-minute rate, but nothing about how many minutes Victoria used.

Taken together, the statements are sufficient. We know that Victoria used 300 non-Sunday minutes in addition to the first 1,000, 200 Sunday minutes, and that the regular per-minute rate is $0.10. From there, we can determine the various discounted rates and calculate her bill for last month. Choice (C) is correct.

8 Explanations: Rate: Challenge

209. C

Explanation: Currently, a certain number of packages can be shipped at a certain price per package for a total of \$120:

$np = 120$

If the price were raised by 1, the number shipped for \$120 would decrease by 6:

$(n - 6)(p + 1) = 120$
$np - 6p + n - 6 = 120$
$np - 6p + n = 126$

Rewrite the first equation so that it can be substituted into the second:

$n = \frac{120}{p}$

Then plug in to the second:

$p(\frac{120}{p}) - 6p + \frac{120}{p} = 126$
$120 - 6p + \frac{120}{p} = 126$
$\frac{120}{p} - 6p = 6$
$120 - 6p^2 = 6p$
$20 - p^2 = p$
$p^2 + p - 20 = 0$
$(p + 5)(p - 4) = 0$
$p = 4$ or $p = -5$

Since the price of shipping a package can't be negative, it must be \$4, choice (C).

210. D

EXPL: Since the rate is given in terms of seconds and the time is given in minutes, you'll have to convert one to the other. It's easier to convert a single number than a fraction, so convert m minutes to seconds: since there are 60 seconds in 1 minute, there are $60m$ seconds in m minutes.

Now, you can use the rate formula. $r = \frac{d}{t}$, and in this case, $r = \frac{f}{s}$ and $t = 60m$. To solve:

$\frac{f}{s} = \frac{d}{60m}$
$d = \frac{60fm}{s}$, choice (D).